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Hopf algebraic structures of $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$ algebras, monopoles and symplectic geometry on 2D manifolds†

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Abstract. From the symplectic structure on the q-deformed manifold S_q^2 of S^2, the classical and quantum q-deformed algebras $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$ of SU(2) are discussed in detail and their Hopf structures are constructed. It is also shown that the symplectic geometry of S^2 describes the properties of monopoles for the SU(2) algebra. The q-(deformed) monopoles are studied from the symplectic geometry on S_q^2, and a relation between generic 2D manifold and algebraic structures is established.

1. Introduction

With neither commutative nor co-commutative Hopf algebraic structures, the 'quantum algebras' [1–4] are commonly referred to the quantum q-deformations of the ordinary Lie algebras. In other words, the q-deformations result in the 'quantization' of the Lie algebras, and in the classical limit $\hbar \to 0$ the 'quantum algebras' reduce to the Lie algebras whose Hopf algebraic structures are trivial.

Nevertheless, it has been pointed out recently [5–7] that the q-deformation and $\hbar$-quantization are in principle independent of each other. It has been found that the q-deformation of a Lie algebra, say, SU(2), may be realized both at the classical and quantum levels, corresponding to the $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$ algebras, respectively. In the approach given in [5, 6], the classical q-deformation of SU(2), $\mathrm{SU}_{q,\hbar\to 0}(2)$, is obtained by deforming a set of classical observables, which are generators of the SU(2) algebra in the sense of the Poisson brackets, and the canonical quantization of the system produces the quantum q-deformed algebra $\mathrm{SU}_{q,\hbar}(2)$ realized by the q-deformed oscillators [8, 9]. On the other hand, in the approach of [7], the classical q-deformation of SU(2) is realized by deforming the symplectic manifold (S^2, ω_0), which is invariant under SU(2), to the corresponding q-deformed symplectic manifold (S_q^2, ω_q), which is of $\mathrm{SU}_q(2)$ symmetry. These two approaches open up a new direction of research on the 'quantum groups' and, in particular, on the q-deformed classical and quantum Hamiltonian systems

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There is, however, an important problem still to be solved in these two approaches. That is, whether the non-trivial Hopf algebraic structure can also be set up in a q-deformed classical mechanical system with a realization of the q-deformed algebra at the classical level. One of the purposes of this paper is to answer this question with the help of the geometric quantization method [10]. To this end, we first recall some results in [7] with respect to obtaining the q-deformation of the SU(2) algebra at the classical level. Then we construct a set of non-commutative operators acting on the sections on the line bundle over the q-deformed symplectic manifold S_q^2 such that they form $SU_{q,\hbar\to 0}(2)$ in Lie brackets and set up a non-trivial Hopf algebra with a parameter q only in such a classical system The key point in constructing such a set of operators is that they are nothing but the so-called prequantization operators of the observables, which form $SU_{q,\hbar\to 0}(2)$ in Poisson brackets. In view of the quantum mechanics, the prequantization is still classical. To emphasize this fact, we make a tiny but crucial distinction between our approach to the so-called prequantization operators and the standard one in the literature [10]. That is, we do not introduce the Planck constant $\hbar$ in these operators before a suitable polarization has been chosen. In other words, the appearance of $\hbar$ is completely quantum characteristic. We use the word 'quantum' in this physical sense only.

By use of the geometric quantization approach, i.e. the polarization, we get the quantum q-deformed algebra $SU_{q,\hbar}(2)$, which has a Hopf algebraic structure with two independent parameters q and $\hbar$, in such a q-deformed quantum mechanical system. The algebra representations and eigenvalue relations are also discussed in detail.

It is found that the line bundle constructed on the S^2 manifold in the symplectic geometry approach to deal with the SU(2) algebra [10] just corresponds to the principal bundle or Hopf bundle in describing monopoles, and the operator that generates the SU(2) algebra is just the angular momentum of a charged particle in a monopole field. Hence by investigating the q-deformed manifold S_q^2 with respect to the $SU_q(2)$ algebra, the q-monopole and the 'angular momentum' symmetry of a particle in a q-monopole field can be studied.

We also show that the deformation from S^2 to S_q^2 is a quasiconformal transformation Also, analogously to the dependence of an algebra and manifold such as SU(2) and S^2, $SU_q(2)$ and S_q^2, we investigate the symplectic geometry of general 2D manifolds and present a relation between 2D manifolds and the corresponding algebra structures.

The paper is organized as follows: in section 2, we give a brief review of the symplectic geometry of SU(2) and $SU_q(2)$. Then, by using the prequantization method we present the classical realization and the Hopf algebraic structure of the algebra $SU_{q,\hbar\to 0}(2)$. In section 3, we investigate the expressions of the generators of $SU_{q,\hbar}(2)$ in the quantum system by taking a suitable polarization, introducing the Planck constant $\hbar$ and setting up the quantum Hopf algebraic structure. In section 4, we discuss the relation between the monopole and the symplectic description of the SU(2) algebra on S^2 and study the q-monopole, using the symplectic description of the $SU_q(2)$ algebra on S_q^2. In section 5 we show that the deformation from the 2D sphere S^2 to S_q^2 is a quasiconformal transformation. We also study the symplectic geometry on a general 2D manifold and give a correspondence between manifolds and algebraic structures. A few concluding remarks are given in section 6. In the appendix we give the relation between approach of [6] and that presented in [7] and this paper, thus showing the relation between the algebra and manifold.

2. Hopf algebraic structure of the $\mathrm{SU}_{q,\hbar\to 0}(2)$ algebra

Let us consider a 2D sphere S^2 such that

$$S_1^2 + S_2^2 + S_3^2 = S_0^2 \tag{1}$$

where S_i, $i = 1, 2, 3$, are three variables and S_0 is a constant, the radius of the sphere.

The symplectic structure can be constructed on the sphere with symplectic form [10]

$$\omega_0 = -\frac{1}{2S_0^2}\sum_{ijk}^{3}\epsilon_{ijk} S_i \,\mathrm{d}S_j \wedge \mathrm{d}S_k \tag{2}$$

and Hamiltonian vector fields of the variables

$$X_{S_i} = \sum_{jk}^{3}\epsilon_{ijk} S_j \frac{\partial}{\partial S_k} \tag{3}$$

The Poisson brackets are

$$[S_i, S_j]_{\mathrm{PB}} = -X_{S_i} S_j = \epsilon_{ijk} S_k. \tag{4}$$

This is just the Lie algebra of SU(2) in the Poisson bracket. Now we consider the q-deformed sphere S_q^2 [7] defined by

$$S_1^2 + S_2^2 + \frac{(\sinh\gamma S_3)^2}{\gamma\sinh\gamma} = S_\gamma^2 \tag{5}$$

where $\gamma = \log q$ is the deformation parameter and S_γ is a constant, i.e. the deformed 'radius'.

The symplectic form and Hamiltonian vector fields for the deformed sphere S_q^2 take the form

$$\omega_\gamma = -\frac{1}{S_\gamma^2}\left(S_1 \mathrm{d}S_2 \wedge \mathrm{d}S_3 + S_2 \mathrm{d}S_3 \wedge \mathrm{d}S_1 + \frac{\tanh\gamma S_3}{\gamma}\mathrm{d}S_1 \wedge \mathrm{d}S_2\right) \tag{6}$$

$$\begin{aligned} X_{S_1} &= S_2\frac{\partial}{\partial S_3} - \frac{\sinh 2\gamma S_3}{2\sinh\gamma}\frac{\partial}{\partial S_2} \\ X_{S_2} &= \frac{\sinh 2\gamma S_3}{2\sinh\gamma}\frac{\partial}{\partial S_1} - S_1\frac{\partial}{\partial S_3} \\ X_{S_3} &= S_1\frac{\partial}{\partial S_2} - S_2\frac{\partial}{\partial S_1} \end{aligned} \tag{7}$$

which, by using relation (5), satisfy the conditions

$$\begin{aligned} &X_{S_i} \rfloor \omega_\gamma = -\mathrm{d}S_i \\ &[X_{S_i}, X_{S_j}] = -X_{[S_i,S_j]_{\mathrm{PB}}} \\ &\omega_\gamma(X_{S_i}, X_{S_j}) = [S_i, S_j]_{\mathrm{PB}} \end{aligned}$$

where ⌋ denotes the left inner product

Therefore, we have the q-deformed algebra $SU_{q,\hbar\to 0}(2)$ in the sense of the Poisson bracket:

$$\begin{aligned}
&[S_1, S_2]_{\rm PB} = \frac{\sinh 2\gamma S_3}{2\sinh\gamma} \\
&[S_2, S_3]_{\rm PB} = S_1 \\
&[S_3, S_1]_{\rm PB} = S_2
\end{aligned} \tag{8}$$

or

$$\begin{aligned}
&[S_+, S_-]_{\rm PB} = -{\rm i}\frac{\sinh 2\gamma S_3}{\sinh\gamma} \\
&[S_3, S_\pm]_{\rm PB} = \mp {\rm i} S_\pm
\end{aligned}$$

where $S_\pm = S_1 \pm {\rm i}S_2$ and ${\rm i} = \sqrt{-1}$. Algebra (8) is isomorphic to the usual 'quantum algebra' $\mathrm{SU}_q(2)$ but it is classically realized as we have not quantized the system at all.

The q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ can also be realized in the Lie bracket of a set of classical operators To do this, let us first construct the line bundle $\tilde{L}$, a bundle similar to the prequantization line bundle except the Planck constant $\hbar$ as is stated above, over the q-deformed symplectic manifold (S_q^2, ω_γ). The curvature of the line bundle is represented by the q-deformed symplectic form ω_γ.

We set up the complex structure on the q-sphere S_q^2 by introducing two open sets:

$$U_\pm = \left\{ x \in S_q^2 \mid S_\gamma \pm \frac{\sinh\gamma S_3}{\sqrt{\gamma\sinh\gamma}} \neq 0 \right\} \tag{9}$$

and two complex functions z_+ and z_- on U_+ and U_-, respectively,

$$z_\pm = (S_1 \mp {\rm i}S_2)\left(S_\gamma \pm \frac{\sinh\gamma S_3}{\sqrt{\gamma\sinh\gamma}}\right)^{-1}. \tag{10}$$

In $U_+ \bigcap U_-$ we have

$$z_+ z_- = 1. \tag{11}$$

From (10) we get expressions of S_i, $i = 1, 2, 3$, in terms of z_+ and z_-,

$$\begin{aligned}
&S_1 = S_\gamma \frac{z_\pm + \overline{z}_\pm}{1 + z_\pm \overline{z}_\pm} \\
&S_2 = \pm {\rm i} S_\gamma \frac{z_\pm - \overline{z}_\pm}{1 + z_\pm \overline{z}_\pm} \\
&\frac{\sinh\gamma S_3}{\sqrt{\gamma\sinh\gamma}} = \pm S_\gamma \frac{1 - z_\pm \overline{z}_\pm}{1 + z_\pm \overline{z}_\pm}.
\end{aligned} \tag{12}$$

The q-deformed symplectic form (6) becomes

$$\begin{aligned}\omega_\gamma \mid U_\pm &= -2\mathrm{i}S_\gamma \left(S_\gamma^2\gamma^2\frac{(1-z_\pm\overline{z}_\pm)^2}{(1+z_\pm\overline{z}_\pm)^2}+\frac{\gamma}{\sinh\gamma}\right)^{-1/2}\frac{\mathrm{d}\overline{z}_\pm\wedge\mathrm{d}z_\pm}{(1+z_\pm\overline{z}_\pm)^2}\\ &= \mathrm{i}Q_\pm\mathrm{d}\overline{z}_\pm\wedge\mathrm{d}z_\pm\end{aligned} \tag{13}$$

where

$$Q_\pm = -2S_\gamma\left(S_\gamma^2\gamma^2\frac{(1-z_\pm\overline{z}_\pm)^2}{(1+z_\pm\overline{z}_\pm)^2}+\frac{\gamma}{\sinh\gamma}\right)^{-1/2}(1+z_\pm\overline{z}_\pm)^{-2}.$$

Since ω_γ is closed, it should locally be exact on the open set U_+ and U_-, i.e

$$\omega_\gamma \mid U_\pm = \mathrm{d}\theta_\pm$$

Here the symplectic 1-forms $\theta_\pm$ are

$$\begin{aligned}\theta_\pm &= \frac{\mathrm{i}}{\gamma z_\pm}\left[\sinh^{-1}\left(S_\gamma\sqrt{\gamma\sinh\gamma}\frac{1-z_\pm\overline{z}_\pm}{1+z_\pm\overline{z}_\pm}\right)-\sinh^{-1}(S_\gamma\sqrt{\gamma\sinh\gamma})\right]\mathrm{d}z_\pm\\ &= \mathrm{i}p_\pm\mathrm{d}z_\pm\end{aligned} \tag{14}$$

where

$$p_\pm = \frac{1}{\gamma z_\pm}\left[\sinh^{-1}\left(S_\gamma\sqrt{\gamma\sinh\gamma}\frac{1-z_\pm\overline{z}_\pm}{1+z_\pm\overline{z}_\pm}\right)-\sinh^{-1}(S_\gamma\sqrt{\gamma\sinh\gamma})\right]. \tag{15}$$

The formulae in (7) now take the forms

$$\begin{aligned}X_{S_1} &= -\mathrm{i}\left(\frac{\gamma}{\sinh\gamma}\right)^{1/2}\frac{\cosh\gamma S_3}{2}\left((z_\pm^2-1)\frac{\partial}{\partial z_\pm}+(1-\overline{z}_\pm^2)\frac{\partial}{\partial\overline{z}_\pm}\right)\\ X_{S_2} &= \pm\left(\frac{\gamma}{\sinh\gamma}\right)^{1/2}\frac{\cosh\gamma S_3}{2}\left((z_\pm^2+1)\frac{\partial}{\partial z_\pm}+(1+\overline{z}_\pm^2)\frac{\partial}{\partial\overline{z}_\pm}\right)\\ X_{S_3} &= \pm\mathrm{i}\left(\overline{z}_\pm\frac{\partial}{\partial\overline{z}_\pm}-z_\pm\frac{\partial}{\partial\overline{z}_\pm}\right)\end{aligned} \tag{16}$$

where S_3 is given by (12) The formulae in (16) also give rise to the $\mathrm{SU}_{q,\hbar\to 0}(2)$ algebra among S_i, $i = 1, 2, 3$, in Poisson brackets (8).

In order to find the classical realization of the algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ in the Lie brackets and to set up the Hopf algebraic structure, we need a linear isomorphism from the 'observables' to the linear operators, which form the algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ in the Lie brackets, on the line bundle $\tilde{L}$ over the symplectic manifold (S_q^2, ω_γ). Such a linear isomorphism can be realized by the following map:

$$f\to\tilde{f} = -\mathrm{i}(X_f-\mathrm{i}\theta(X_f))+f \tag{17}$$

for some functions f on S_q^2, where $\tilde{f}$ is the linear operator expression of f. This map, however, does not work for S_1 and S_2 as they have to obey the constraint (5). Therefore, instead of directly finding such a map from S_i to $\tilde{S}_i$, $i = 1, 2, 3$, we should work with some basic quantities.

Let us rewrite expressions (12) in terms of variables $z_\pm$ and $p_\pm$ by using (15),

$$
\begin{aligned}
S_1 &= \frac{1}{\sqrt{\gamma \sinh \gamma}} \Big(\cosh(\tfrac{1}{2}\gamma z_\pm p_\pm) z_\pm \sinh(\tfrac{1}{2}\gamma(z_\pm p_\pm + 2b)) \\
&\quad - \cosh(\tfrac{1}{2}\gamma(z_\pm p_\pm + 2b)) \frac{1}{z_\pm} \sinh(\tfrac{1}{2}\gamma z_\pm p_\pm) \Big) \\
S_2 &= \frac{\pm \mathrm{i}}{\sqrt{\gamma \sinh \gamma}} \Big(\cosh(\tfrac{1}{2}\gamma z_\pm p_\pm) z_\pm \sinh(\tfrac{1}{2}\gamma 2(z_\pm p_\pm + 2b)) \\
&\quad + \cosh(\tfrac{1}{2}\gamma(z_\pm p_\pm + 2b)) \frac{1}{z_\pm} \sinh(\tfrac{1}{2}\gamma z_\pm p_\pm) \Big) \\
S_3 &= \pm(z_\pm p_\pm + b).
\end{aligned}
\tag{18}
$$

Here, for convenience, we denote

$$\sinh \gamma b = S_\gamma \sqrt{\gamma \sinh \gamma}. \tag{19}$$

From formula (13), we have the Hamiltonian vector fields of z and p,

$$
\begin{aligned}
X_{z_\pm} &= \mathrm{i} Q_\pm^{-1} \frac{\partial}{\partial \bar{z}_\pm} \\
X_{p_\pm} &= -\mathrm{i} \frac{\partial}{\partial z_\pm} + \mathrm{i} Q_\pm^{-1} \frac{\partial p_\pm}{\partial z_\pm} \frac{\partial}{\partial \bar{z}_\pm}.
\end{aligned}
\tag{20}
$$

Using the map given above, we have their operator representations:

$$
\begin{aligned}
\tilde{z}_\pm &= Q_\pm^{-1} \frac{\partial}{\partial \bar{z}_\pm} + z_\pm \\
\tilde{p}_\pm &= -\frac{\partial}{\partial z_\pm} + Q_\pm^{-1} \frac{\partial p_\pm}{\partial z_\pm} \frac{\partial}{\partial \bar{z}_\pm}.
\end{aligned}
\tag{21}
$$

It is easy to check that the commutator of $\tilde{z}$ and $\tilde{p}$ is

$$[\tilde{z}_\pm, \tilde{p}_\pm] = 1 \tag{22}$$

where the relation

$$\frac{\partial p_\pm}{\partial \bar{z}_\pm} = Q_\pm$$

has been used.

By means of the formulae in (21), the operators with respect to (18) can be expressed as

$$\begin{aligned}\tilde{S}_1 &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\Big(\cosh(\tfrac{1}{2}\gamma\tilde{z}_\pm\tilde{p}_\pm)\tilde{z}_\pm\sinh(\tfrac{1}{2}\gamma(\tilde{z}_\pm\tilde{p}_\pm+2b))\\ &\quad -\cosh(\tfrac{1}{2}\gamma(\tilde{z}_\pm\tilde{p}_\pm+2b))\frac{\gamma\tilde{p}_\pm}{2}\sum_{n=0}^{\infty}((2n+1)!)^{-1}(\tfrac{1}{2}\gamma\tilde{z}_\pm\tilde{p}_\pm)^{2n}\Big)\\ \tilde{S}_2 &= \frac{\pm i}{\sqrt{\gamma\sinh\gamma}}\Big(\cosh(\tfrac{1}{2}\gamma\tilde{z}_\pm\tilde{p}_\pm)\tilde{z}_\pm\sinh(\tfrac{1}{2}\gamma(\tilde{z}_\pm\tilde{p}_\pm+2b))\\ &\quad +\cosh(\tfrac{1}{2}\gamma(\tilde{z}_\pm\tilde{p}_\pm+2b))\frac{\gamma\tilde{p}_\pm}{2}\sum_{n=0}^{\infty}((2n+1)!)^{-1}(\tfrac{1}{2}\gamma\tilde{z}_\pm\tilde{p}_\pm)^{2n}\Big)\\ \tilde{S}_3 &= \pm(\tilde{z}_\pm\tilde{p}_\pm+b).\end{aligned}\tag{23}$$

It is straightforward to check that they form the $SU_{q,\hbar\to 0}(2)$ algebra in Lie commutators:

$$\begin{aligned}[\tilde{S}_1,\tilde{S}_2] &= i\frac{\sinh 2\gamma\tilde{S}_3}{2\gamma}\\ [\tilde{S}_2,\tilde{S}_3] &= i\tilde{S}_1\\ [\tilde{S}_3,\tilde{S}_1] &= i\tilde{S}_2\end{aligned}\tag{24}$$

or

$$[\tilde{S}_+,\tilde{S}_-] = \frac{\sinh 2\gamma\tilde{S}_3}{\gamma}\qquad [\tilde{S}_3,\tilde{S}_\pm] = \pm\tilde{S}_\pm$$

where $\tilde{S}_\pm = \tilde{S}_1 \pm i\tilde{S}_2$. Algebra (24) is isomorphic to the usual $SU_q(2)$ algebra up to a constant factor $\sinh\gamma/\gamma$ that can be removed by redefining $\tilde{S}_1$ and $\tilde{S}_2$

It should be emphasized again that so far this q-deformed symplectic system has not been quantized at all. In other words, the algebraic relations (24) are still a classical realization of the q-deformed algebra $SU_{q,\hbar\to 0}(2)$, although the generators are non-commutative operators

Having these operators in (23) and (24), we can define the Hopf algebraic structure [1] of $SU_{q,\hbar\to 0}(2)$ in its universal enveloping algebra,

$$\begin{aligned}&\Delta(\tilde{S}_3) = \tilde{S}_3\otimes 1 + 1\otimes\tilde{S}_3\\ &\Delta(\tilde{S}_\pm) = \tilde{S}_\pm\otimes e^{\gamma\tilde{S}_3} + e^{-\gamma\tilde{S}_3}\otimes\tilde{S}_\pm\\ &\varepsilon(1) = 1\qquad \varepsilon(\tilde{S}_\pm) = \varepsilon(\tilde{S}_3) = 0\\ &\eta(\tilde{S}_\pm) = -e^{\gamma\tilde{S}_3}\tilde{S}_\pm e^{-\gamma\tilde{S}_3}\qquad \eta(\tilde{S}_3) = -\tilde{S}_3\end{aligned}\tag{25}$$

where Δ is the co-product, η is the antipode and ε is the co-unit. Hence in the sense of these formulae in (25), $SU_{q,\hbar\to 0}(2)$ is a classical Hopf algebra which is neither commutative nor co-commutative.

Furthermore, the universal R-matrices satisfying the 'quantum' Yang–Baxter relation with one parameter q can also be given.

3. The geometric quantization of the q-deformed symplectic system

The geometric quantization of a classical system is described by the quantization line bundle and polarization [10], and in this case the quantization line bundle L exists if and only if $h^{-1}\omega_\gamma$ defines an integral de Rham cohomology class, i.e. the de Rham cohomology class $[-h^{-1}\omega_\gamma]$ of $-h^{-1}\omega_\gamma$ should be integrable. This leads to

$$-h^{-1}\int_{S_q^2}\omega_\gamma = 2j \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

Hence we have

$$S_\gamma = \frac{\sinh\gamma j\hbar}{\sqrt{\gamma\sinh\gamma}} \tag{26}$$

where h is the Planck constant. Comparing equations (19) and (26) we see that here b takes integral and half-integral times of $\hbar$. Moreover, equation (26) is different from the case of the sphere S^2 where the condition that the de Rham cohomology class $[-h^{-1}\omega_0]$ is integrable leads to $S_0 = j\hbar$.

As for a suitable polarization, let us consider the linear frame fields $X_{z_\pm}$,

$$X_{z_\pm} = \mathrm{i}Q_\pm^{-1}\frac{\partial}{\partial\overline{z}_\pm}.$$

For each $x \in U_+ \cap U_-$, we have

$$X_{z_-} = -z_+^{-2}X_{z_+}.$$

Hence, X_{z_+} and X_{z_-} span a complex distribution F on S_q^2 and F is a polarization of the symplectic manifold (S_q^2, ω_γ). Moreover,

$$\mathrm{i}\omega_\gamma(X_{z_\pm}, \overline{X}_{z_\pm}) = \frac{1}{2S_\gamma}\left(S_\gamma^2\gamma^2\frac{(1-z_\pm\overline{z}_\pm)^2}{(1+z_\pm\overline{z}_\pm)^2} + \frac{\gamma}{\sinh\gamma}\right)^{1/2}(1+z_\pm\overline{z}_\pm)^2 > 0.$$

This means that F is a complete strongly admissible positive polarization of (S_q^2, ω_γ).

Let λ be the local section of $L \mid U_\pm$ such that

$$\nabla\lambda_\pm = -\mathrm{i}\hbar^{-1}\theta_\pm \otimes \lambda_\pm \tag{27}$$

In $U_+ \cap U_-$, we have

$$-\mathrm{i}\hbar^{-1}(\theta_+ - \theta_-) = \mathrm{d}\log z_-^{2j} \tag{28}$$

where equation (26) has been taken into account. From (28), it is clear that the transition function z_-^{2j} is globally defined and single-valued on $U_+ \cap U_-$ as $2j$ is an integer. We may assume that λ_+ and λ_- are so normalized that

$$\lambda_+ = z_-^{2j}\lambda_-. \tag{29}$$

From (27), it is obvious that the sections λ_+ and λ_- are covariantly constant along the polarization F,

$$\nabla_{X_{z_\pm}}\lambda_\pm = \langle\nabla\lambda_\pm, X_{z_\pm}\rangle = 0.$$

Consequently each covariantly constant section λ of L can be expressed as follows:

$$\Psi \mid U_\pm = \psi_\pm(z_\pm)\lambda_\pm$$

where $\psi_\pm(z_\pm)$ is the holomorphic function of $z_\pm$. Taking (28) and (11) into account, on $U_+ \cap U_-$ we have the relation

$$\psi_+(z_+) = z_+^{2j}\psi_-(z_+^{-1}).$$

This implies that ψ_+ and ψ_- are polynomials of degree at most $2j$ in the variable. Therefore, the section Ψ_j with respect to a certain number j may be expressed as

$$\Psi_j = \sum_{m=-j}^{j} \Psi_{jm}$$

where

$$\Psi_{jm} \mid U_+ = c_{jm} z_+^{j-m}\lambda_+$$
$$\Psi_{jm} \mid U_- = c_{jm} z_-^{j+m}\lambda_- \qquad -j \le m \le j \tag{30}$$

and c_{jm} are some constants depending on the normalization. As the connection $\theta_\pm$ invariant Hermitian metric on bundle L satisfies the equation

$$\mathrm{d}\langle\lambda_\pm, \lambda_\pm\rangle = -\mathrm{i}\hbar^{-1}\langle\lambda_\pm, \lambda_\pm\rangle(\theta_\pm - \overline{\theta}_\pm)$$

the normalization constants c_{jm} here are so chosen that

$$\int_{S_q^2}\langle\Psi_{jm}, \Psi_{jm}\rangle\omega_\gamma = 1.$$

From (16), it is obvious that the generators S_1 and S_2 are not polarization-preserving functions. Hence, to get their quantum operator expressions one has to use the method of BKS kernels [10] and which includes complicated and tedious calculations. This is why we use the classical operator expressions of $\tilde{p}$ and $\tilde{z}$ to construct the classical operators with respect to the S_i in the previous section.

To obtain the quantum operator expressions of S_i, $i = 1, 2, 3$, we start from the quantum operators of p and z first. For a general polarization-preserving function f, its quantum operator may be written as

$$\hat{f} = -\mathrm{i}\hbar X_f - \theta(X_f) + f - \tfrac{1}{2}\mathrm{i}\hbar a$$

where the first three terms, corresponding to the classical operator $\tilde{f}$, correspond to the prequantization operator of f and a is determined by the equation

$$[X_f, X_{z_\pm}] = aX_{z_\pm}.$$

Here, we put the Planck constant $\hbar$ in the quantum map $f \to \hat{f}$. This is because this map really stands for the physical quantization as the polarization has been introduced here and the section space of the line bundle L is now restricted to the quantum Hilbert space.

For the polarization-preserving functions p and z, from (20) we have

$$[X_{p_\pm}, X_{z_\pm}] = 0.$$

Therefore, their quantum operators are

$$\hat{p}_\pm = -\hbar \frac{\partial}{\partial z_\pm}$$
$$\hat{z}_\pm = z_\pm \tag{31}$$

where the terms with derivative $\partial/\partial\bar{z}$ have been omitted as the section space is covariantly constant along the polarization F. So the quantum commutator of $\hat{p}$ and $\hat{z}$ is simply

$$[\hat{z}_\pm, \hat{p}_\pm] = \hbar. \tag{32}$$

Having (18) and (31), we obtain the quantum operators with suitable ordering as follows:

$$\begin{aligned}
\hat{S}_1 &= \frac{1}{\sqrt{\gamma \sinh \gamma}} \left[\cosh\left(\frac{\gamma\hbar}{2} z_\pm \frac{\partial}{\partial z_\pm}\right) z_\pm \sinh\left(\frac{\gamma\hbar}{2}\left(-z_\pm \frac{\partial}{\partial z_\pm} + 2j\right)\right) \right. \\
&\quad \left. + \cosh\left(\frac{\gamma\hbar}{2}\left(-z_\pm \frac{\partial}{\partial z_\pm} + 2j\right)\right) \frac{1}{z_\pm} \sinh\left(\frac{\gamma\hbar}{2} z_\pm \frac{\partial}{\partial z_\pm}\right)\right] \\
\hat{S}_2 &= \frac{\pm i}{\sqrt{\gamma \sinh \gamma}} \left[\cosh\left(\frac{\gamma\hbar}{2} z_\pm \frac{\partial}{\partial z_\pm}\right) z_\pm \sinh\left(\frac{\gamma\hbar}{2}\left(-z_\pm \frac{\partial}{\partial z_\pm} + 2j\right)\right) \right. \\
&\quad \left. - \cosh\left(\frac{\gamma\hbar}{2}\left(-z_\pm \frac{\partial}{\partial z_\pm} + 2j\right)\right) \frac{1}{z_\pm} \sinh\left(\frac{\gamma\hbar}{2} z_\pm \frac{\partial}{\partial z_\pm}\right)\right] \\
\hat{S}_3 &= \pm\hbar\left(-z_\pm \frac{\partial}{\partial z_\pm} + j\right).
\end{aligned} \tag{33}$$

They give rise to the quantum q-deformed algebra of SU(2), $\mathrm{SU}_{q,\hbar}(2)$, as follows:

$$[\hat{S}_+, \hat{S}_-] = \frac{\sinh(\gamma\hbar)}{\gamma} \frac{\sinh 2\gamma \hat{S}_3}{\sinh \gamma}$$
$$[\hat{S}_3, \hat{S}_\pm] = \pm\hbar \hat{S}_\pm. \tag{34}$$

Here, instead of taking the Planck constant to be unity, we deliberately let the $\hbar$ remain in the formulae. Algebra (34) coincides with the usual $\mathrm{SU}_q(2)$ algebra up to some removable but significant constant factors about $\hbar$.

It is clear that the 'quantum algebra' in a quantum mechanical system may be characterized by two independent parameters $q = e^{\gamma}$ and $\hbar$ as was pointed out in [5–7]. The former characterizes the deformation of the system either at the classical level or at the quantum level and gives rise to the q-deformation of the system as well as the Lie algebra, while the latter determines the quantization of the classical system that leads to the q-deformed algebra at the physically quantum level. In other words, when γ approaches zero, the 'quantum algebra' becomes the usual Lie algebra for either the undeformed classical or quantum systems

By using the quantum operators (33) and the representation space (30), the eigenvalue relations of $\hat{S}_i$ are now

$$\begin{aligned}\hat{S}_1\Psi_{jm} &= \frac{1}{\sqrt{\gamma\sinh\gamma}}[\cosh(\tfrac{1}{2}\gamma(j-m+1)\hbar)\sinh(\tfrac{1}{2}\gamma(j+m)\hbar)c^{-1}_{jm-1}c_{jm}\Psi_{jm-1} \\ &\quad + \cosh(\tfrac{1}{2}\gamma(j+m+1)\hbar)\sinh(\tfrac{1}{2}\gamma(j-m)\hbar)c^{-1}_{jm+1}c_{jm}\Psi_{jm+1}] \\ \hat{S}_2\Psi_{jm} &= \frac{i}{\sqrt{\gamma\sinh\gamma}}[\cosh(\tfrac{1}{2}\gamma(j-m+1)\hbar)\sinh(\tfrac{1}{2}\gamma(j+m)\hbar)c^{-1}_{jm-1}c_{jm}\Psi_{jm-1} \\ &\quad - \cosh(\tfrac{1}{2}\gamma(j+m+1)\hbar)\sinh(\tfrac{1}{2}\gamma(j-m)\hbar)c^{-1}_{jm+1}c_{jm}\Psi_{jm+1}] \\ \hat{S}_3\Psi_{jm} &= m\hbar\Psi_{jm}.\end{aligned}$$

or

$$\begin{aligned}\hat{S}_+\Psi_{jm} &= \frac{2}{\sqrt{\gamma\sinh\gamma}}\cosh(\tfrac{1}{2}\gamma(j+m+1)\hbar)\sinh(\tfrac{1}{2}\gamma(j-m)\hbar)c^{-1}_{jm+1}c_{jm}\Psi_{jm+1} \\ \hat{S}_-\Psi_{jm} &= \frac{2}{\sqrt{\gamma\sinh\gamma}}\cosh(\tfrac{1}{2}\gamma(j-m+1)\hbar)\sinh(\tfrac{1}{2}\gamma(j+m)\hbar)c^{-1}_{jm-1}c_{jm}\Psi_{jm-1} \qquad (35) \\ \hat{S}_3\Psi_{jm} &= m\hbar\Psi_{jm}.\end{aligned}$$

From formula (35) we may define an operator $\hat{C}_q$ and find its eigenvalue:

$$\begin{aligned}\hat{C}_q &= \hat{S}_+\hat{S}_- + \frac{\sinh^2\gamma(\hat{S}_3 - \frac{1}{2}\hbar)}{\gamma\sinh\gamma} = \hat{S}_-\hat{S}_+ + \frac{\sinh^2\gamma(\hat{S}_3 + \frac{1}{2}\hbar)}{\gamma\sinh\gamma} \\ \hat{C}_q\Psi_{jm} &= \frac{\sinh^2\gamma(j+\frac{1}{2})\hbar}{\gamma\sinh\gamma}\Psi_{jm}.\end{aligned} \qquad (36)$$

$\hat{C}_q$ is the Casimir operator of $SU_{q,\hbar}(2)$ The formulae in (36) are the same as the usual results [2, 8, 11] except for the constant factor $\sinh\gamma/\gamma$.

Similar to the case of the sphere S^2 where one has the total angular momentum operator $\hat{J}$ and its eigenvalue [10]

$$\hat{J}^2 = S_+\hat{S}_- + \hat{S}_3(\hat{S}_3 - \hbar) = \hat{S}_-\hat{S}_+ + \hat{S}_3(\hat{S}_3+\hbar) \qquad \hat{J}^2\Psi_{jm} = j(j+1)\hbar^2\Psi_{jm}$$

here we have corresponding relations from equation (36):

$$\begin{aligned}\hat{J}^2_q &= \hat{S}_+\hat{S}_- + \frac{\sinh\gamma\hat{S}_3\sinh\gamma(\hat{S}_3-\hbar)}{\gamma\sinh\gamma} = \hat{S}_-\hat{S}_+ + \frac{\sinh\gamma\hat{S}_3\sinh\gamma(\hat{S}_3+\hbar)}{\gamma\sinh\gamma} \\ \hat{J}^2_q\Psi_{jm} &= \frac{\sinh(\gamma j\hbar)\sinh(\gamma(j+1)\hbar)}{\gamma\sinh\gamma}\Psi_{jm}.\end{aligned} \qquad (37)$$

Consequently we may consider $(\sinh(\gamma j\hbar)\sinh(\gamma(j+1)\hbar))/(\gamma\sinh\gamma)$ as the eigenvalue of the square of the q-deformed total 'angular momentum' operator.

In fact, from equation (33) we directly have the relations between $\hat{S}_+$, $\hat{S}_-$ and $\hat{S}_3$,

$$\frac{\sinh^2\gamma(j+\frac{1}{2})\hbar}{\gamma\sinh\gamma}=\hat{S}_+\hat{S}_-+\frac{\sinh^2\gamma(\hat{S}_3-\frac{1}{2}\hbar)}{\gamma\sinh\gamma}=\hat{S}_-\hat{S}_++\frac{\sinh^2\gamma(\hat{S}_3+\frac{1}{2}\hbar)}{\gamma\sinh\gamma}$$

or

$$\frac{\sinh(\gamma j\hbar)\sinh(\gamma(j+1)\hbar)}{\gamma\sinh\gamma}=\hat{S}_+\hat{S}_-+\frac{\sinh\gamma\hat{S}_3\sinh\gamma(\hat{S}_3-\hbar)}{\gamma\sinh\gamma}$$
$$=\hat{S}_-\hat{S}_++\frac{\sinh\gamma\hat{S}_3\sinh\gamma(\hat{S}_3+\hbar)}{\gamma\sinh\gamma}$$

which define a 'quantum' version of the q-deformed sphere S_q^2.

As for the Hopf algebraic structure of the quantum q-deformed algebra $\mathrm{SU}_{q,\hbar}(2)$, we may define the co-product, co-unit and antipode operators by

$$\begin{aligned}&\Delta(\hat{S}_3)=\hat{S}_3\otimes 1+1\otimes\hat{S}_3\\&\Delta(\hat{S}_\pm)=\hat{S}_\pm\otimes e^{\gamma\hat{S}_3}+e^{-\gamma\hat{S}_3}\otimes\hat{S}_\pm\\&\varepsilon(1)=1\qquad\varepsilon(\hat{S}_\pm)=\varepsilon(\hat{S}_3)=0\\&\eta(\hat{S}_\pm)=-e^{\gamma\hat{S}_3}\hat{S}_\pm e^{-\gamma\hat{S}_3}\qquad\eta(\hat{S}_3)=-\hat{S}_3.\end{aligned}\tag{38}$$

Therefore, as it has neither commutative nor co-commutative Hopf algebraic structures, algebra (34) is in fact the 'quantum algebra' at the physically quantum level. Furthermore, the universal R-matrices satisfying the 'quantum' Yang–Baxter relation with both parameters q and $\hbar$ can be given readily [11].

4. Monopoles and q-deformed monopoles

The usual magnetic monopoles are geometrically described by the non-trivial principal U_1 bundles on the manifold S^2 [12]. It has been shown [13] that the U_1 principal bundle $P(S^2,U_1)$ is the Hopf bundle S^3/Z_D, where, taking the electric charge and magnetic charge to be unity, Z_D is a cyclic group of order $D=2$. We find that the magnetic monopole may also be described by the symplectic geometry on S^2 at the classical (prequantization) level. The prequantization line bundle on S^2 corresponds to the Hopf bundle and the symplectic form corresponds to the curvature of the bundle. Hence the symplectic 1-form just represents the monopole potential

To see this, let us introduce two open sets V_+ and V_- on the sphere S^2 defined by (1),

$$V_\pm=\{x\in S^2\mid S_0\pm S_3(x)\neq 0\}$$

and two complex functions w_+ and w_- on V_+ and V_-, respectively,

$$w_\pm=\frac{S_1\mp iS_2}{S_0\pm S_3}.\tag{39}$$

In terms of the complex coordinates w_+ and w_-, the symplectic form (2) becomes

$$\omega_0 \mid V_\pm = -2iS_0 \frac{d\overline{w}_\pm \wedge dw_\pm}{(1+w_\pm\overline{w}_\pm)^2}.$$

The prequantization line bundle on S^2 exists if and only if

$$-\frac{1}{2\pi}\int_{S^2}\omega_0 = D$$

where D is an integer. This gives rise to the fact that $S_0 = \frac{1}{2}D$ Here, unlike the analysis in [10], at the prequantization level we would not put in the Planck constant, for the reason given in section 1.

The restriction of ω_0 to the open sets V_+ and V_- are exact·

$$\omega_0|V_\pm = dA_\pm$$

where

$$A_\pm = -iS_0\frac{\bar{w_\pm}dw_\pm - w_\pm d\bar{w_\pm}}{1+w_\pm\bar{w_\pm}}. \tag{40}$$

The symplectic 1-form $A_\pm$ is just the usual monopole potential. Rewritting it in spherical coordinates we get its familiar form [12, 13]:

$$A_\pm = -S_0(\pm 1 - \cos\theta)d\psi. \tag{41}$$

In $V_+ \cap V_-$ we have

$$-i(A_+ - A_-) = 2iS_0 d\psi = d\log e^{iD\psi}$$

Hence the transition function is $e^{iD\psi}$ and the integer D is just the mapping degree [13].

Now we calculate the prequantization operators Expressing (40) in terms of coordinates S_1, S_2 and S_3 we have

$$A = \frac{S_2 dS_1 - S_1 dS_2}{S_0 + S_3}$$

i.e.

$$A_1 = \frac{S_2}{S_0+S_3} \qquad A_2 = -\frac{S_1}{S_0+S_3} \qquad A_3 = 0. \tag{42}$$

Using the prequantization map (17) we obtain

$$\tilde{\boldsymbol{S}} = \boldsymbol{r}\times(\boldsymbol{p}-\boldsymbol{A}) + \boldsymbol{r} \qquad i = 1,2,3 \tag{43}$$

where the vectors $\boldsymbol{r} = (S_1, S_2, S_3)$, $\tilde{\boldsymbol{S}} = (\tilde{S}_1, \tilde{S}_2, \tilde{S}_3)$, $\boldsymbol{p} = (-i\partial/\partial S_1, -i\partial/\partial S_2, -i\partial/\partial S_3)$ and $\boldsymbol{A} = (A_1, A_2, A_3)$.

Equation (43) is just the usual angular momentum of a charged particle moving in a monopole field [14]. The algebra generated by $\tilde{S}_1$,$\tilde{S}_2$ and $\tilde{S}_3$ is isomorphic to Poisson algebra (4). They satisfy the usual angular momentum commutator relations.

It is manifest now that the symplectic form ω_0 on S^2 corresponds to the field strength of monopole and the symplectic 1-form A is the monopole potential The prequantization operators S_i which generate the SU(2) algebra represent the angular momentum of a particle in a monopole field. Therefore it is straightforward to discuss the q-(deformed) monopole according to the q-deformed symplectic system. Hence the q-monopole is described by the symplectic manifold (S_q^2, ω_γ) defined by equations (5) and (6). To match the form in (40), we write the symplectic 1-form, i e. the potential of the q-monopole, in a symmetric way,

$$A_{q\pm} = i(p_\pm \mathrm{d}z_\pm - \bar{p_\pm}\mathrm{d}\bar{z_\pm}) \tag{44}$$

where $p_\pm$ is defined by equation (15)

Formula (44) may also be expressed as

$$A_{q\pm} = -\left(\pm\frac{\sinh^{-1}(S_\gamma\sqrt{\gamma\sinh\gamma})}{\gamma} - \frac{\sinh^{-1}(S_\gamma\sqrt{\gamma\sinh\gamma}\cos\theta)}{\gamma}\right)\mathrm{d}\psi$$

or

$$A_q = (b - S_3)\left(S_\gamma^2 - \frac{(\sinh\gamma S_3)^2}{\gamma\sinh\gamma}\right)^{-1}(S_2\mathrm{d}S_1 - S_1\mathrm{d}S_2)$$

i.e.

$$A_{q1} = \frac{b - S_3}{S_1^2 + S_2^2}S_2 \qquad A_{q2} = -\frac{b - S_3}{S_1^2 + S_2^2}S_1 \qquad A_{q3} = 0$$

where b is defined by equation (18). One may compare these with equations (41) and (42).

By using equations (13) and (40), similar to the consideration in section 2, we may obtain the prequantization operators of S_1, S_2 and S_3, which again generate the algebra (24). That is, the angular momentum of a particle moving in the q-monopole field is of $\mathrm{SU}_q(2)$ symmetry.

5. Symplectic geometry and algebraic structures on 2D manifolds

From the previous sections we see that the algebra SU(2) and $\mathrm{SU}_q(2)$ are connected by a q-deformation and that both may have classical and quantum realizations. As the algebra SU(2) corresponds to the manifold S^2 and $\mathrm{SU}_q(2)$ to the manifold S_q^2 in the sense of the symplectic approach, the q-deformation just deforms the manifold S^2 to S_q^2 and hence the symplectic structure on S^2 to that of S_q^2. In fact, by using a similar approach, we can give a symplectic structure to general 2D manifolds including S^2 and S_q^2.

Let us consider a 2D surface defined by

$$f(S_1, S_2, S_3) = 0 \tag{45}$$

where f is a smooth function of variables S_i, $i = 1, 2, 3$ A symplectic structure on such an even-dimensional manifold may be constructed with symplectic form

$$\omega = A\mathrm{d}S_2 \wedge \mathrm{d}S_3 + B\mathrm{d}S_3 \wedge \mathrm{d}S_1 + C\mathrm{d}S_1 \wedge \mathrm{d}S_2 \tag{46}$$

and Hamiltonian vector fields

$$\begin{aligned} X_{S_1} &= \alpha\left(\frac{\partial f}{\partial S_2}\frac{\partial}{\partial S_3} - \frac{\partial f}{\partial S_3}\frac{\partial}{\partial S_2}\right) \\ X_{S_2} &= \alpha\left(\frac{\partial f}{\partial S_3}\frac{\partial}{\partial S_1} - \frac{\partial f}{\partial S_1}\frac{\partial}{\partial S_3}\right) \\ X_{S_3} &= \alpha\left(\frac{\partial f}{\partial S_1}\frac{\partial}{\partial S_2} - \frac{\partial f}{\partial S_2}\frac{\partial}{\partial S_1}\right) \end{aligned} \tag{47}$$

by assuming that A, B and C satisfy

$$-\alpha\left(A\frac{\partial f}{\partial S_1} + B\frac{\partial f}{\partial S_2} + C\frac{\partial f}{\partial S_3}\right) = 1 \tag{48}$$

where A, B and C are smooth functions of S_i and α is a constant. In terms of (45) ω and X_{S_i} satisfy the following relations:

$$\begin{aligned} &X_{S_i}\rfloor\omega = -\mathrm{d}S_i \\ &[X_{S_i}, X_{S_j}] = -X_{[S_i,S_j]_{\mathrm{PB}}} \\ &\omega(X_{S_i}, X_{S_j}) = [S_i, S_j]_{\mathrm{PB}}. \end{aligned} \tag{49}$$

In view of the similar method used in [10], this symplectic structure is uniquely determined by the constraint equation (45). From equations (47), we obtain the Poisson brackets of the S_i.

$$[S_i, S_j]_{\mathrm{PB}} = \alpha\epsilon_{ijk}\frac{\partial f}{\partial S_k}. \tag{50}$$

Since α is a constant, (50) shows that the algebraic structure is uniquely determined by the manifold defined by (45). Hence such a symplectic structure establishes a correspondence between the 2D manifold and the algebraic relation.

For instance, for the sphere S^2,

$$f = S_1^2 + S_2^2 + S_3^2 - S_0^2$$

From (48), we have

$$-\alpha(2AS_1 + 2BS_2 + 2CS_3) = 1.$$

Accounting to the condition $f = 0$, one may simply take

$$\alpha = \frac{1}{2} \qquad A = -\frac{S_1}{S_0^2} \qquad B = -\frac{S_2}{S_0^2} \qquad C = -\frac{S_3}{S_0^2}.$$

Then, by using (46), (47) and (50), we get the original formulae (2), (3) and (4)

Similarly, for the q-deformed sphere S_q^2,

$$f = S_1^2 + S_2^2 + \frac{(\sinh\gamma S_3)^2}{\gamma\sinh\gamma} - S_\gamma^2$$

We have

$$\alpha = \frac{1}{2} \qquad A = -\frac{S_1}{S_\gamma^2} \qquad B = -\frac{S_2}{S_\gamma^2} \qquad C = -\frac{\tanh\gamma S_3}{\gamma S_\gamma^2}$$

which lead to the previous q-deformed symplectic form (6), the q-deformed Hamiltonian vectors (7) and the q-deformed algebra $SU_{q,\hbar\to 0}(2)$ in (8).

For the one-sheet hyperboloid,

$$f = \frac{S_1^2}{a^2} + \frac{S_2^2}{b^2} - \frac{S_3^2}{c^2} - 1$$

where a, b and c are constants, we have

$$[S_1, S_2]_{PB} = -\frac{S_3}{c^2} \qquad [S_2, S_3]_{PB} = \frac{S_1}{a^2} \qquad [S_3, S_1]_{PB} = \frac{S_2}{b^2}.$$

This is the well-known algebra SU(1, 1). Similarly, we may deform the hyperboloid to get the q-deformed algebra $SU_{q,\hbar\to 0}(1,1)$

For another example one may easily find that the elliptic paraboloid

$$\frac{S_1^2}{a^2} + \frac{S_2^2}{b^2} - 2S_3 = h_0 \tag{51}$$

and q-deformed elliptic paraboloid

$$\frac{S_1^2}{a^2} + \frac{S_2^2}{b^2} - \frac{\sinh(2\gamma S_3)}{\gamma\cosh\gamma} = h_\gamma \tag{52}$$

with a, b, h_0 and h_γ constants, correspond to the simple harmonic oscillator algebra $\mathcal{H}(4)$ [15] and q-deformed algebra of $\mathcal{H}(4)$, $\mathcal{H}_q(4)$ [16], respectively, where the constants h_0 and h_γ correspond to Casimir operators of the algebras From equations (50) and (51) it is obvious that the two manifolds with respect to algebra $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$ are not compact However, corresponding to the algebras SU(2) and $SU_q(2)$, the manifolds defined by equations (1) and (5) are compact. This explains why the algebra transformations from SU(2) to $\mathcal{H}(4)$ [17] and $SU_q(2)$ to $\mathcal{H}_q(4)$ [18] are of singularities.

It is clear now that as a 2D manifold implies a certain algebraic relation, a continuous deformation of the manifold may result in the continuous deformation of the algebraic relation Here it is worthwhile to investigate what kind of deformation is from the S^2 to the S_q^2 manifold

To find out, let us compare complex coordinates $w_\pm$ and $z_\pm$ of manifolds S^2 and S_q^2. From equations (39) and (10) we have

$$z_\pm = \frac{2S_0 w_\pm}{1+w_\pm\overline{w}_\pm}\frac{\sqrt{\gamma\sinh\gamma}}{S_\gamma\sqrt{\gamma\sinh\gamma}+\sinh[\gamma S_0(1-w_\pm\overline{w}_\pm)/(1+w_\pm\overline{w}_\pm)]}$$

$$\overline{z}_\pm = \frac{2S_0 \overline{w}_\pm}{1+w_\pm\overline{w}_\pm}\frac{\sqrt{\gamma\sinh\gamma}}{S_\gamma\sqrt{\gamma\sinh\gamma}+\sinh[\gamma S_0(1-w_\pm\overline{w}_\pm)/(1+w_\pm\overline{w}_\pm)]}. \tag{53}$$

From these transformation relations, it is straightforward to calculate the so-called the Beltrami coefficient [19] with respect to these transformations:

$$\mu = \frac{\partial z_\pm}{\partial \overline{w}_\pm}\left(\frac{\partial z_\pm}{\partial w_\pm}\right)^{-1}$$
$$= w_\pm^2\left[\frac{2S_\gamma\gamma}{1+w_\pm\overline{w}_\pm}\cosh\left(\frac{S_\gamma\gamma(1-w_\pm\overline{w}_\pm)}{1+w_\pm\overline{w}_\pm}\right)\right.$$
$$\left. - \sinh\left(\frac{S_\gamma\gamma(1-w_\pm\overline{w}_\pm)}{1+w_\pm\overline{w}_\pm}\right) - \sqrt{\gamma\sinh\gamma}\right]$$
$$\times\left[\frac{2S_\gamma\gamma}{1+w_\pm\overline{w}_\pm}\cosh\left(\frac{S_\gamma\gamma(1-w_\pm\overline{w}_\pm)}{1+w_\pm\overline{w}_\pm}\right)\right.$$
$$\left. + \sinh\left(\frac{S_\gamma\gamma(1-w_\pm\overline{w}_\pm)}{1+w_\pm\overline{w}_\pm}\right) + \sqrt{\gamma\sinh\gamma}\right]^{-1} \quad (54)$$

Obviously, $|\mu| < 1$ if q is real. This means that if q is real the deformation from S^2 to S_q^2 in each coordinate piece is a quasi-conformal transformation. It is this quasi-conformal transformation and its 'quantum' version that give rise to the classical and quantum q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$, respectively.

6. Remarks

We have realized the q-deformed algebra $\mathrm{SU}_q(2)$ both at the classical mechanical and quantum mechanical levels, based upon deforming the underlying symplectic structures. We have shown that the change from the SU(2) to $\mathrm{SU}_q(2)$ algebra is related to a deformation from the sphere S^2 to the q-sphere S_q^2 and the deformation is a quasi-conformal transformation when q is real.

An important result in this paper is the setting of the Hopf algebraic structure of the classical q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ with a set of non-commutative operators defined on the line bundle over the q-deformed symplectic manifold, i.e. the q-sphere S_q^2. This may shed new light on the 'quantum groups' in general. One should note that in this paper the classical and quantum operators of S_i are obtained in terms of the operators of $p_\pm$ and $z_\pm$ and a suitable ordering. Nevertheless, in principle one may also find the quantum operators through direct calculation by means of BKS kernel methods and prequantization operators, i.e. the classical operators that give rise to the classical deformed and undeformed algebras in Lie brackets are also given [10].

Instead of setting the Hopf algebraic structure at the classical mechanical level with the operator representations via the prequantization approach, it is also possible to realize this algebraic structure by means of deformation theory [20]. That is, we may find a suitable multiplication operation, similar to the Moyal product [21], $*_q$, of functions on the q-sphere S_q^2 such that the classical q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ may be realized in the sense of the Moyal-like bracket consistent with the $*_q$-product.

The geometric property of the q-sphere is also intriguing. Classically, it is most like a q-deformed top. After the $\hbar$ quantization, the constraint equation for the q-sphere becomes something like a non-commutative geometric realization of the Casimir operator of the quantum q-deformed algebra $SU_{q,\hbar}(2)$. In fact, this Casimir operator should be taken as the Hamiltonian, up to some constant as zero-point energy, of the quantum q-deformed top. Its quantum behaviour should be the q-deformation of an ordinary quantum top. Further investigation is needed in this direction.

The usual so-called 'quantum algebra' is also studied by the method of non-commutative C^*-algebras [4, 22]. In particular, in [23] the $SU_q(2)$ algebra is discussed in terms of the 'quantum sphere' and the differential calculus on it It is of interest to investigate the relations between the 'quantum sphere' and the quantum version of the q-sphere S_q^2 and so on

Above all, with the correspondence between the constraint equation defining the underlying symplectic structure of the symmetry and the Casimir operator of the algebra, the approach to the classical and quantum q-deformations of the SU(2) Lie algebra presented in [7] and in this paper should be able to be generalized to generic Lie algebras All these and relevant subjects as well as the further symplectic geometry description of monopoles and q-monopoles are under investigation.

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Appendix

From the discussions in section 5, we see that in the symplectic geometry approach the algebra structure with three generators is tightly related to a 2D manifold generally defined by the Casimir operator, although the symplectic structure with symplectic form (46) may be of different forms That is, once the manifold (45) is given, an algebra is determinated by the formula (50) only up to an algebra isomorphism.

In this paper and in [7] we have shown that the $SU_q(2)$ algebra with q-deformed symplectic form ω_γ is related to the manifold S_q^2. Nevertheless, in the approach presented in [6] the $SU_q(2)$ algebra is obtained by deforming the usual SU(2) generator S_i to S_i' without deforming the symplectic structure. Hence we would like to know if in such an approach there still exists a hidden S_q^2 manifold for $SU_q(2)$.

In [6] the classical SU(2) algebra is described by the symplectic form

$$\Omega = -\mathrm{i}\sum_{i=1}^{2} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i \tag{A1}$$

and observables (generators of SU(2))

$$S_+ = z_1\bar{z}_2 \qquad S_- = z_2\bar{z}_1 \qquad S_3 = \tfrac{1}{2}(z_2\bar{z}_2 - z_1\bar{z}_1). \tag{A2}$$

With respect to (A1) one has

$$[z_i, \bar{z}_j]_{\rm PB} = -{\rm i}\delta_{ij} \qquad [{\rm others}]_{\rm PB} = 0. \tag{A3}$$

In terms of (A3) $S_\pm$ and S_3 give rise to the Poisson algebra (4) of SU(2).

The classical $SU_q(2)$ algebra is obtained by setting the new observables

$$S'_+ = z_1\bar{z}_2{}' \qquad S'_- = z_2\bar{z}_1{}' \qquad S'_3 = S_3 \tag{A4}$$

with

$$z'_i = \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\sinh(\gamma z_i\bar{z}_i)}{z_i\bar{z}_i}z_i \qquad \bar{z}_i{}' = \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\sinh(\gamma z_i\bar{z}_i)}{z_i\bar{z}_i}\bar{z}_i.$$

Then, using the original symplectic form (A1) and hence the relations (A3), $S'_\pm$ and S'_3 constitute the $SU_q(2)$ algebra.

Now for the classical SU(2) algebra let us suppose

$$S_+S_- + S_3^2 = S_0^2 \tag{A5}$$

in accordance with equation (1), where S_0 is the constant radius of S^2. Hence from (A2) we may write $z_1\bar{z}_1 = S_0 + S_3$, and $z_2\bar{z}_2 = S_0 - S_3$. Substituting these into equation (A4) we obtain

$$S'_\pm = \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\sinh\gamma(S_0 \pm S'_3)}{S_0 \pm S'_3}S_\pm.$$

From equation (A5) we immediately have

$$S'_+S'_- + \frac{(\sinh\gamma S'_3)^2}{\gamma\sinh\gamma} = \frac{(\sinh\gamma S_0)^2}{\gamma\sinh\gamma}. \tag{A6}$$

Equation (A6) is just the manifold S_q^2 defined by equation (5) with $S_\gamma = (\sinh\gamma S_0)^2/\gamma\sinh\gamma$. Therefore the deformed sphere S_q^2 is still hidden in the deformed observables $S'_\pm$ and S'_3 which generate the $SU_q(2)$ algebra.

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